

Inversion of perturbation series

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We investigate the inversion of perturbation series and its resummation, and prove that it is related to a recently developed parametric perturbation theory. Results for some illustrative examples show that in some cases series reversion may improve the accuracy of the results.

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I. INTRODUCTION

Perturbation theory yields the solution of a problem in the form of a power series of a properly chosen model (or dummy) parameter. When the convergence radius of this series is too small for the physical application, or the series converges too slowly, it is customary to resort to a resummation method that produces an approximant that improves the results. There are many approaches for that purpose; among them we mention Padé approximants, Borel summation, algebraic approximants, continued fractions, nonlinear transformations of the expansion parameter, etc. [1, 2, 3].

The purpose of this paper is to investigate the application of a well-known mathematical method, named series reversion or series inversion [4, 5, 6], to perturbation theory. In Sec. II we outline the main ideas of the approach, in Sec. III we show the connection between the method of series reversion and a recently developed parametric perturbation theory [7, 8, 9]. In Sec. IV we compare the accuracy of resummation of the direct and inverse series for some illustrative examples. Finally, in Sec. V we draw some conclusions about the usefulness of the approach.

II. INVERSION OF SERIES

Suppose that we are trying to estimate accurate values of an unknown function $E(g)$ from a few available coefficients of its power series

$$E(g) = \sum_{j=0}^{\infty} E_j g^j \quad (1)$$

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When the convergence radius of this series is too small for the physical application, or the series converges too slowly, it is customary to resort to a resummation method that produces an approximant $E \approx A_E^{[N]}(g)$ from the partial sum of order N : $S_N = E_0 + E_1g + \dots + E_Ng^N$ [1, 2, 3].

It is always possible to invert the series (1) and obtain g in terms of $\Delta E = E - E_0$: [4, 5, 6]

$$g = \Delta E \sum_{j=0}^{\infty} G_j \Delta E^j = \frac{\Delta E}{E_1} - \frac{E_2 \Delta E^2}{E_1^3} + \frac{(2E_2^2 - E_1 E_3) \Delta E^3}{E_1^5} + \dots \quad (2)$$

If we apply a resummation method to this series we obtain an approximation of the form $g \approx A_g^{[N]}(\Delta E)$. This strategy may give more accurate results than the former one if the radius of convergence or the region of utility of the series (2) is greater than that of the direct expansion (1). However, even in this favorable situation we are paying the price of having the inverse of the desired function.

In order to get some motivation for the use of the inverse series, consider the function

$$E = \sqrt{1+g} = 1 + \frac{1}{2}g - \frac{1}{8}g^2 + \dots \quad (3)$$

that is real for all $g > -1$. The Taylor series about $g = 0$ converges only for $|g| < 1$ because of the branch point at $g = -1$. However, the inverse series $g = 2\Delta E + \Delta E^2$ converges for all ΔE suggesting that in some cases it is convenient to use the latter instead of the former.

III. PARAMETRIC PERTURBATION THEORY

In what follows we show that the inversion of series is related to a recently developed parametric perturbation theory [7, 8, 9]. If we define the parameter $\rho = \Delta E/E_1$ then

$$g = \rho \sum_{j=0}^{\infty} G_j E_1^{j+1} \rho^j = \rho - \frac{E_2 \rho^2}{E_1} + \frac{(2E_2^2 - E_1 E_3) \rho^3}{E_1^2} + \dots \quad (4)$$

Application of a resummation method gives an expression for g in terms of the parameter ρ :

$$g \approx A_g^{[N]}(E_1 \rho) \quad (5)$$

which together with

$$E = E_0 + E_1 \rho \quad (6)$$

define an approximate parametric representation of $E(g)$. These equations are the basis of the so called parametric perturbation theory developed recently by Amore [7, 8, 9] from a principle of absolute simplicity.

More precisely, in parametric perturbation theory one introduces an approximant $g = A^{[N]}(\rho)$ into the direct expansion (1) and choose the approximant coefficients so that $E = b_0 + b_1 \rho + [\rho^{N+1}]$. Since the approximant is constructed so that $A^{[N]}(\rho) \approx \rho$ for $\rho \ll 1$ [7, 8, 9] then we realize that $b_0 = E_0$ and $b_1 = E_1$, and conclude that parametric perturbation theory is basically the inversion of the perturbation series.

In order to see the connection clearer consider the trivial example (3) again. If we substitute $\rho + c_2 \rho^2 + \dots + c_N \rho^N$ for g into the series (3) and choose the coefficients c_j in order to remove the terms of order greater than one, we obtain

$c_2 = 1/4$ and $c_j = 0$, $j > 2$. The resulting expressions $E = 1 + \rho/2$ and $g = \rho + \rho^2/4$ exactly agree with the inverse series shown above.

In some cases one easily estimates a range of utility of the parametric representation of $E(g)$. For example, suppose that we know that $dE/dg < 0$ (> 0), then $E_1 < 0$ (> 0). In both cases the range of utility of the approximant $g \approx A^{[N]}(\rho)$ is determined by the conditions $\rho > 0$ and $dg/d\rho > 0$.

IV. EXAMPLES

As an illustrative example we consider the integral

$$E(g) = \int_0^\infty e^{-x^2 - gx^4} dx. \quad (7)$$

The formal expansion

$$E(g) = \frac{\sqrt{\pi}}{2} \left(1 - \frac{3g}{4} + \frac{105g^2}{32} - \frac{3465g^3}{128} + \frac{675675g^4}{2048} - \frac{43648605g^5}{8192} + \dots \right) \quad (8)$$

is known to diverge for all values of g [2] which is reflected by the form of the expansion coefficients $E_n = (-1)^n \Gamma(2n + 1/2)/[2(n!)]$. We can rewrite the inverse expansion

$$g(\Delta E) = -\frac{8\Delta E}{3\sqrt{\pi}} + \frac{280\Delta E^2}{9\pi} - \dots \quad (9)$$

in terms of the parameter $\rho = -3\sqrt{\pi}\Delta E/8$ as

$$g(\rho) = \rho + \frac{35\rho^2}{8} + \frac{35\rho^3}{16} + \frac{17675\rho^4}{256} - \frac{263095\rho^5}{256} + \dots \quad (10)$$

In this case we have

$$E(\rho) = \frac{\sqrt{\pi}}{2} - \frac{3\sqrt{\pi}\rho}{8} \quad (11)$$

The pair of equations for $g(\rho)$ and $E(\rho)$ are a parametric representation of the function $E(g)$.

For simplicity we restrict to series of order five with the purpose of illustrating some ways of constructing the approximants. In order to determine the range of applicability of the ρ -series we take into consideration that $dg/d\rho > 0$ for all $\rho > 0$. Therefore, we assume that it may be reasonable to use the parametric representation for all $0 < \rho < \rho_m$, where ρ_m is the smallest positive root of $dg/d\rho = 0$. In the case of the partial sum of order five (10) we have $\rho_m = 0.1659$ that leads to $g_m = g(\rho_m) = 0.2194$. Table I shows that the parametric representation embodied in equations (10) and (11) yields better results than the direct g -power series (8), at least for all $g < g_m$.

We can improve the accuracy of the results by means of resummation methods. For simplicity we try Padé approximants [1, 2] built from the g -series and ρ -series of order five shown above. Since $E(g \rightarrow \infty) = 0$ we choose the [2/3] Padé approximant

$$E \approx \frac{2\sqrt{\pi}(115460139g^2 + 28532448g + 1163200)}{141105195g^3 + 534788100g^2 + 117619392g + 4652800} \quad (12)$$

The [2/3] Padé approximant for the inverse series

$$g = \frac{16\rho(219707\rho + 15640)}{26152805\rho^3 - 11137140\rho^2 + 2420512\rho + 250240} \quad (13)$$

presents a maximum at $\rho_m = 0.360$ where $g_m = 0.607$. Table II shows that the Padé approximant for the direct series is more accurate than the one for the inverse series.

The $[3/2]$ Padé approximant

$$g = \frac{\rho (15904 + 318204\rho + 747223\rho^2)}{15904 + 248624\rho - 375297\rho^2} \quad (14)$$

exhibits a pole at $\rho_0 = 0.721$ and we can therefore use this parametric representation for all $0 < g < \infty$ when $0 < \rho < \rho_0$. The main limitation of this approach is that $E(g \rightarrow \infty) = E_0 + E_1\rho_0 = 0.407$. Table III shows that the inverse series is slightly more accurate for small g but leads to a wrong limit. On the other hand, the direct series approaches the correct limit too fast as $1/g$.

The approximants so far used do not take into consideration the asymptotic behavior $E(g) \sim g^{-1/4}$ as $g \rightarrow \infty$. In the case of the direct series we may calculate $[N/N+1](g)$ Padé approximants for $E(g)^4$ and then use $[N/N+1](g)^{1/4}$ as a reasonable approximation; for example

$$E \approx \frac{\sqrt{\pi}}{2} \left(\frac{116949g^2 + 27216g + 1060}{218277g^3 + 190647g^2 + 30396g + 1060} \right)^{1/4}. \quad (15)$$

We do not know how to obtain reasonable approximants for the inverse series; for that reason we try Amore's approach $g(\rho) = \rho[N/N+1](\rho)^5$; for example,

$$g = \frac{-1048576\rho(24078737127\rho^2 + 4884321032\rho + 161747680)^5}{(240656732281\rho^3 - 321691293064\rho^2 - 75884668992\rho - 2587962880)^5} \quad (16)$$

constructed from the ρ -series of order six. One realizes that this is the parametric perturbation approach proposed by Amore [7, 8, 9] for this particular case. In the neighbourhood of the pole $\rho = \rho_0$ closest to origin this approximant behaves as $g \approx K_0(\rho - \rho_0)^{-5}$ so that $E(g) \approx E_0 + E_1\rho_0 + E_1K_0^{1/5}g^{-1/5}$. We appreciate that this parametric representation tends to a wrong limit (because $E_0 + E_1\rho_0 \neq 0$ in general) and in a wrong way. Table IV shows that these approximants are more accurate than the previous ones, as expected, and that the direct series is clearly better than the inverse one. Notice that the parametric representation gives the wrong limit $E(g \rightarrow \infty) = 0.861$. Besides, remember that we need perturbation coefficients of order 5 and 6 of the direct and reverse series, respectively, in the construction of such approximants.

Following the same philosophy we have also tried $[3/2]$ approximants for the reverse series

$$g = \frac{\rho (583627586759\rho^3 + 767002386296\rho^2 + 44364247040\rho - 265333760)^5}{32768 (290961289397\rho^2 + 5574551760\rho - 33166720)^5} \quad (17)$$

Table V shows that this parametric representation yields better results than the preceding one, but it also leads to a wrong limit: $E(g \rightarrow \infty) = 0.883$.

The polylogarithm function

$$Li_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - z} dt \quad (18)$$

appears in several fields of theoretical physics, for example, in the Bose-Einstein and Fermi-Dirac distributions [10]. The Taylor expansion of this function about $z = 0$ yields the series

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \quad (19)$$

In this case the inverse series gives more accurate results than the direct one and, consequently, we draw the same conclusion regarding the Padé approximants constructed from them. Table VI shows that the Padé approximant $[5/6](\rho)$ for the former series gives considerably more accurate results than the $[6/7](g)$ Padé approximant for the latter one.

The formal Taylor series about $g = 0$ for the function

$$E(g) = \int_0^\infty \frac{e^{-x}}{1+gx} dx \quad (20)$$

is divergent as shown by its coefficients $E_n = (-1)^n n!$. The asymptotic behavior at $g \gg 1$ is given by

$$E(g) \approx \frac{\ln(g) - \gamma}{g} + \frac{\ln(g) - \gamma + 1}{g^2} + \dots \quad (21)$$

where $\gamma = 0.5772156649\dots$ is Euler's constant. In this case the inverse series is more accurate than the direct one, and exactly the same situation takes place for their corresponding Padé approximants.

As another illustrative example consider the function $E(g)$ defined by $g = Ee^{-E}$. The power series $E(g) = g + g^2 + 3g^2/2 + \dots$ converges for $g < e^{-1}$ which is reflected by the form of the expansion coefficients $E_n = n^{n-1}/n!$, $n \geq 1$. On the other hand, the inverse series $g = E - E^2 + \dots + (-1)^{n-1} E^n / (n-1)! + \dots$ converges for all E and is therefore preferable for practical applications. However, it is not always true that the inverse series have greater convergence radius than the direct one; simply consider the function $E(g) = ge^{-g}$, where the role of the variables has been reversed. In this case parametric perturbation theory will perform poorer than standard perturbation theory.

V. CONCLUSIONS

In this paper we investigated the usefulness of the inversion of perturbation series and its resummation. Our simple examples show that in some cases it is convenient to resort to the inverse series, but in others the straightforward perturbation series is expected to provide better results. We have also proved that the recently proposed parametric perturbation theory [7, 8, 9], based on the principle of absolute simplicity, consists of a convenient modification and resummation of the inverse series. Therefore, in some cases this approach will not perform better than well-known resummation methods on the direct series. In addition to it, parametric perturbation theory exhibits two disadvantages: producing the inverse of the desired function, and the difficulty of deriving the correct asymptotic limit and behaviour when it is known.

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TABLE I: Series expansions of order five for the integral (7)

g	ρ	Eq.(8)	Eq.(10)	Exact
0.04	0.227268254	0.8630223905	0.8632172917	0.8632281022
0.08	0.2211979138	0.8358839701	0.8447690498	0.8449941504
0.12	0.213865426	0.7500155805	0.8285524745	0.8296883134
0.16	0.2043179764	0.4525374722	0.7504228564	0.8164228001
0.2	0.1891655735	-0.3655376234	0.7604942069	0.8046805576

TABLE II: Padé approximants $[2/3]$ for the integral (7)

g	ρ	Eq.(12)	Eq.(13)	Exact
0.1	0.07419851329	0.836884189	0.8369093852	0.8370429277
0.2	0.124755604	0.803286224	0.8033055939	0.8046805576
0.3	0.166889096	0.7760554974	0.7753007175	0.7800434542
0.4	0.2069906611	0.7523625938	0.7486464025	0.7600358198
0.5	0.2515059286	0.7310124346	0.719058431	0.7431554088
0.6	0.392980645	0.7113938594	0.6250244038	0.7285439429

TABLE III: Padé approximant $[2/3]$ for the g -series and $[3/2]$ for the inverse series of the integral (7)

g	ρ	Eq.(12)	Eq.(14)	Exact
0.1	0.07404520814	0.836884189	0.8370112825	0.8370429277
1	0.3079042644	0.6446831837	0.6815721382	0.6842134278
10	0.6043806547	0.2142628036	0.4845131183	0.4609804743
100	0.7061454902	0.02801405374	0.4168730654	0.2772884009
1000	0.7196727352	0.0028904009730	0.4078819088	0.1594808649
10000	0.7210732433	0.0002899613860	0.4069510328	0.09033502245

TABLE IV: Improved Padé approximants $[2/3]$ for the integral (7)

g	ρ	Eq.(15)	Eq.(16)	Exact
0.1	0.07356402668	0.8369445716	0.8373311095	0.8370429277
1	0.2892446121	0.6715809835	0.6939746529	0.6842134278
10	0.5828537448	0.4198251659	0.4988214137	0.4609804743
100	0.03723484095	0.2393860646	0.8614780364	0.2772884009
1000	0.03737319161	0.1348101326	0.8613860789	0.1594808649
10000	0.03737319161	0.07582023369	0.8613860789	0.09033502245

TABLE V: Improved Padé approximant $[3/2]$ for the inverse series of the integral (7)

g	ρ	Eq.(17)
0.1	0.07404345742	0.8370124462
1	0.3170281954	0.6755077332
10	0.8091513837	0.3484081181
100	0.00545012921	0.882604387
1000	0.00545012921	0.882604387
10000	0.00545012921	0.882604387

TABLE VI: Values of $Li_{3/2}(g)$ for g close to 1 obtained from Padé approximants on the direct and inverse series. Exact figures are underlined for comparison.

g	$[6/7](g)$	$[5/6](\rho)$	Exact
0.999999	<u>2.380740506</u>	<u>2.608744256</u>	2.608831900
0.99999	<u>2.380591082</u>	<u>2.601153011</u>	2.601179942
0.9999	<u>2.379099267</u>	<u>2.577063920</u>	2.577071427
0.999	<u>2.364418183</u>	<u>2.501706883</u>	2.501708465
0.99	<u>2.237103024</u>	<u>2.271659944</u>	2.271660077
0.9	<u>1.614336255</u>	<u>1.614438528</u>	1.614438529